Student Paper

Tooth Geometry Calculations of the Involute Gears via Singular Perturbation Methods

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Abstract

The tooth geometry calculations of the involute gears involve the evaluations of involute function $\varepsilon = \tan\phi - \phi$ or $\varepsilon = \text{inv}\phi$, and inverse involute function $\phi = \text{inv}^{-1}(\varepsilon)$. Usually, the inverse function is calculated by using an extensive tabulation of (ϕ, invϕ) which is given in many textbooks and reference books. In this paper, asymptotic series solutions and the corresponding estimated error expressions of the inverse involute function are derived by singular perturbation methods. The computation of the inverse involute function using these asymptotic series can be easily performed by using a pocket calculator. It is shown that the solution by two term asymptotic inverse involute series $\phi = \text{inv}^{-1}(\varepsilon) = (3\varepsilon)^{1/3} - 2\varepsilon/5$, with maximum error less than 1.0% of the angle $\phi(< 45^\circ)$, is almost as accurate as that obtained by linear interpolation from an extensive table of (ϕ, invϕ); and the solution by four term asymptotic series $\phi = \text{inv}^{-1}(\varepsilon)$, with maximum error less than 0.0018%, which is much more accurate than the interpolation method, is almost same as the exact value. Sample applications of these asymptotic series in the tooth geometry calculations of the involute gears are given.

I. Introduction

The involute curve is most widely used for gear tooth shape. In Fig.1, the involute curve BC is generated with respect to the base circle with radius r_b with the property that length AP equals the arc length AB. We can derive the parametric expression for the involute function for BC as follows [5–8].

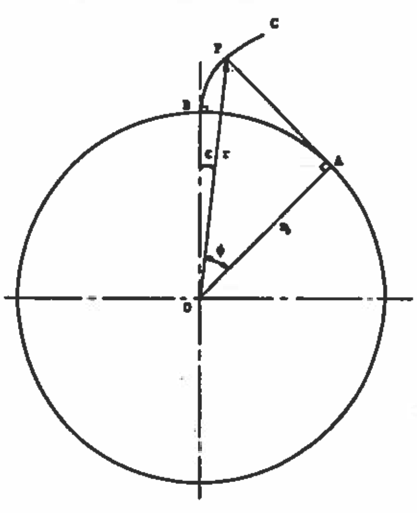

Figure 1 Base circle and involute curve.

$$\begin{cases} r = \dfrac{r_b}{\cos\phi} \\ \varepsilon = \text{inv}\phi = \tan\phi - \phi. \end{cases} \tag{1}$$

If the variable angle ϕ is known, invϕ can be readily determined. But sometimes the problem is how to find ϕ when $\varepsilon = \text{inv}\phi$ is known. This question arises quite often in tooth geometry calculations in gearing. Obviously, no exact solution for $\phi = \text{inv}^{-1}(\varepsilon)$ can be expressed explicitly in term of elementary functions of ε. One might solve for ϕ iteratively using computers, but this is not an efficient way. Some mechanism textbooks, such as Ref.[5–6], give an extensive table of (ϕ, invϕ), from which $\text{inv}^{-1}(\varepsilon)$ can be interpolated. But such a table has its disvantages: the values can not be manipulated mathematically; interpolation is usually required in order to get expected result; and finally, it is difficult to estimate the errors.

Based on the above considerations, the explicit expressions for ϕ in terms of $\varepsilon = \text{inv}\phi$ are derived by singular perturbation methods.

II. Formulation

The involute pressure angle ϕ is usually less than $\pi/4$ when the involute curve is used as the tooth curve of spur gears, therefore, $\text{inv}\phi < \text{inv}(\pi/4) = \tan(\pi/4) - \pi/4 \sim 0.215$. For convenience, let $\varepsilon = \text{inv}\phi$ and $x = \phi$, then Eq.(1) becomes

$$\tan x - x = \varepsilon, \qquad \varepsilon \ll 1. \tag{2}$$

Since $\tan x - x = f(x)$ is an odd function of x, we can consider only the case when $\varepsilon > 0$, but the result will be valid for $\varepsilon < 0$.

Let $y_1 = \tan x, y_2 = x + \varepsilon$; then the solution of Eq.(2) for given ε is the intersection of the curves of y_1 and y_2 as shown in Fig.2 . There are infinite solutions of x corresponding to a given

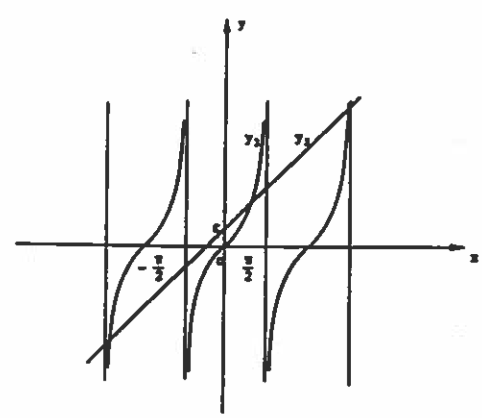

Figure 2 Graphs of $y_1 = \tan x$ and $y_2 = x + \varepsilon$.

Presented at the 10th Applied Mechanisms Conference, Dec., 1987, New Orleans, LA.

value of ε in Eq.(2). We only consider the solutions which lie within $x \in (-\pi/2, \pi/2)$. But the following method can be easily extended to interval $x \in (-\infty, \infty)$. When $|x| < \pi/2$, the Taylor series expansion of $\tan x$ is

$$\begin{aligned} \tan x = & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 \\ & + \frac{62}{2835}x^9 + \cdots + \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}x^{2n-1} + \cdots, \end{aligned} \tag{3}$$

where B_n is Bernoulli's constant. Substituting Eq.(3) into Eq.(2), we get

$$\frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots = \varepsilon. \tag{4}$$

According to singular perturbation theorems [1-4], let

$$x(\varepsilon) \sim \sum_{n=1}^{\infty} \delta_n(\varepsilon) x_n, \qquad \text{as} \quad \varepsilon \to 0^+, \tag{5}$$

where $\delta_{n+1}(\varepsilon) \ll \delta_n(\varepsilon)$, as $\varepsilon \to 0^+$. We then try to find the first four terms of the asymptotic series of x in Eq.(5). Writing Eq.(5) explicitly,

$$x(\varepsilon) \sim \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_4 + O(\delta_5), \tag{6}$$

where δ_i are functions of ε, and both δ_i and x_i need to be determined. Substituting Eq.(6) into Eq.(4), we get

$$\begin{aligned} & \frac{1}{3}[\delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_4 + O(\delta_5)]^3 + \frac{2}{15}[\delta_1 x_1 + \cdots]^5 \\ & \quad + \frac{17}{315}[\delta_1 x_1 + \cdots]^7 + \frac{62}{2835}[\delta_1 x_1 + \cdots]^9 + \cdots \sim \varepsilon, \end{aligned} \tag{7}$$

or

$$\begin{aligned} & \frac{1}{3}\{[(\delta_1 x_1 + \delta_2 x_2)^3 + 3(\delta_1 x_1 + \delta_2 x_2)^2(\delta_3 x_3) + O(\delta_1 \delta_3^2)] \\ & \qquad + [3(\delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3)^2(\delta_4 x_4)] + [O(\delta_1 \delta_4^2)]\} \\ & + \frac{2}{15}\{[(\delta_1 x_1 + \delta_2 x_2)^5 + 5(\delta_1 x_1 + \delta_2 x_2)^4(\delta_3 x_3) + O(\delta_1^3 \delta_3^2)] \\ & \qquad + 5(\delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3)^4(\delta_4 x_4) + O(\delta_1^3 \delta_4^2)\} \\ & + \frac{17}{315}\{[(\delta_1 x_1 + \delta_2 x_2)^7 + 7(\delta_1 x_1 + \delta_2 x_2)^6(\delta_3 x_3) \\ & \qquad + O(\delta_1^5 \delta_3^2)] + O(\delta_1^6 \delta_4)\} \\ & + \frac{62}{2835}\{(\delta_1 x_1)^9 + O(\delta_1^8 \delta_2)\} + O(\delta_1^{11}) \sim \varepsilon. \end{aligned} \tag{8}$$

For convenience, we define the neglected term to be $O(h)$. expanding Eq.(8) again, we get

$$\begin{aligned} & \frac{1}{3}\{[(\delta_1 x_1)^3 + 3(\delta_1 x_1)^2(\delta_2 x_2) + 3(\delta_1 x_1)(\delta_2 x_2)^2 + (\delta_2 x_2)^3] \\ & \qquad + 3[(\delta_1 x_1)^2 + 2(\delta_1 x_1)2(\delta_2 x_2) + (\delta_2 x_2)^2](\delta_3 x_3) \\ & \qquad + 3(\delta_1 x_1)^2(\delta_4 x_4) + O(h)\} \\ & + \frac{2}{15}\{[(\delta_1 x_1)^5 + 5(\delta_1 x_1)^4(\delta_2 x_2) + 10(\delta_1 x_1)^3(\delta_2 x_2)^2 + O(h)] \\ & \qquad + 5(\delta_1 x_1)^4(\delta_3 x_3) + O(h)\} \\ & + \frac{17}{315}\{[(\delta_1 x_1)^7 + 7(\delta_1 x_1)^6(\delta_2 x_2) + O(h)] \\ & \qquad + 7(\delta_1 x_1)^6(\delta_4 x_4) + O(h)\} \\ & + \frac{62}{2835}\{(\delta_1 x_1)^9 + O(h)\} + O(h) \sim \varepsilon. \end{aligned} \tag{9}$$

Comparing the order on the both sides of Eq.(9) by using the dominant balance method, we obtain the following results [1-4]:

(i) $\mathrm{ord}(\delta_1^3) = \mathrm{ord}(\varepsilon)$:

$$\text{then} \quad \delta_1^3 = \varepsilon, \qquad \text{or} \quad \delta_1 = \varepsilon^{1/3}.$$

Setting the coefficient of δ_1^3 equal to the coefficient of ε in Eq.(9), we get

$$\frac{1}{3}x_1^3 = 1$$

$$\text{or} \qquad x_1 = 3^{2k\pi i/3}, \qquad k = 0, 1, 2$$

According to Fig.2, we only consider the real root, hence

$$x_1 = 3^{1/3}$$

(ii) $\mathrm{ord}(\delta_1^2 \delta_2) = \mathrm{ord}(\delta_1^5) = \mathrm{ord}(\varepsilon^{5/3})$:

$$\text{then} \quad \delta_1^2 \delta_2 = \delta_1^5, \quad \text{or} \quad \delta_2 = \delta_1^3 = \varepsilon.$$

Because the coefficient of $\varepsilon^{5/3}$ in Eq.(9) is zero, we get

$$\frac{1}{3}(3x_1^2 x_2) + \frac{2}{15}x_1^5 = 0,$$

$$\text{or} \qquad x_2 = -\frac{2}{15}x_1^3 = -\frac{2}{5}.$$

(iii) $\mathrm{ord}(\delta_1 \delta_2^2) = \mathrm{ord}(\delta_1^2 \delta_3) = \mathrm{ord}(\delta_1^4 \delta_2) = \mathrm{ord}(\delta_1^7) = \mathrm{ord}(\varepsilon^{7/3})$:

$$\text{then} \quad \delta_1^2 \delta_3 = \delta_1^7, \quad \text{or} \quad \delta_3 = \delta_1^5 = \varepsilon^{5/3}.$$

Setting coefficient of $\varepsilon^{7/3}$ in Eq.(9) to zero, we get

$$\frac{1}{3}(3x_1 x_2^2 + 3x_1^2 x_3) + \frac{2}{15}(5x_1^4 x_2) + \frac{17}{315}x_1^7 = 0,$$

and hence

$$\begin{aligned} x_3 & = -(x_1^{-1} x_2^2 + \frac{2}{3}x_1^2 x_2 + \frac{17}{315}x_1^6) \\ & = \frac{9}{175}3^{2/3} \end{aligned}$$

(iv) $\mathrm{ord}(\delta_2^3) = \mathrm{ord}(\delta_1 \delta_2 \delta_3) = \mathrm{ord}(\delta_1^3 \delta_2^2) = \mathrm{ord}(\delta_1^4 \delta_3) = \mathrm{ord}(\delta_1^6 \delta_2) = \mathrm{ord}(\delta_1^2 \delta_4) = \mathrm{ord}(\delta_1^9) = \mathrm{ord}(\varepsilon^3)$:

$$\text{then} \quad \delta_1^2 \delta_4 = \delta_1^9, \quad \text{or} \quad \delta_4 = \delta_1^7 = \varepsilon^{7/3}.$$

Following the same procedure as shown above, we can get

$$\begin{aligned} & \frac{1}{3}(x_2^3 + 6x_1 x_2 x_3 + 3x_1^2 x_4) + \frac{2}{15}(10x_1^3 x_2^2 + 5x_1^4 x_3) \\ & \qquad + \frac{17}{315}(7x_1^6 x_2) + \frac{62}{2835}x_1^9 = 0. \end{aligned}$$

Substituting $x_1 = 3^{1/3}, x_2 = -2/5, x_3 = 9 \bullet 3^{2/3}/175$ into above equation, finally, we obtain

$$x_4 = -\frac{2}{175}3^{1/3},$$

Eq.(6) now gives the following asymptotic series for $x = \mathrm{inv}^{-1}(\varepsilon)$.

$$x(\varepsilon) \sim 3^{1/3}\varepsilon^{1/3} - \frac{2}{5}\varepsilon + \frac{9}{175}3^{2/3}\varepsilon^{5/3} - \frac{2}{175}3^{1/3}\varepsilon^{7/3} + \cdot \tag{10}$$

$$\text{or} \qquad x(\varepsilon) \sim \sum_{n=1}^{\infty} a_n \varepsilon^{(2n-1)/3}, \tag{11}$$

where $\qquad a_1 = 3^{1/3}, a_2 = -\frac{2}{5}, a_3 = \frac{9}{175}3^{2/3}, a_4 = -\frac{2}{175}3^{1/3}$, etc.

For the asymptotic series (11),

$$x(\varepsilon) - \sum_{n=1}^{N} a_n \varepsilon^{(2n-1)/3} \ll a_N \varepsilon^{(2N-1)/3}, \quad \text{as} \quad \varepsilon \to 0^+, \forall N \tag{12}$$

or, alternatively

Table 1

Comparision of $x(\varepsilon) = \text{inv}^{-1}(\varepsilon)$ and estimated errors $E(\varepsilon)$ with their exact values

ε	x_{exact}	$x_4(\varepsilon)$	E_4		$x_2(\varepsilon)$	E_2	
			E_{exact}	$E_4(\varepsilon)$		E_{exact}	$E_2(\varepsilon)$
$1.77 \cdot 10^{-6}$	1°	1°	0	0	1°	0	0
$2.22 \cdot 10^{-4}$	5°	5°	0	0	4° 59′ 59.98″	0.02″	0.02″
$1.79 \cdot 10^{-3}$	10°	10°	0	0	9° 59′ 59.42″	0.58″	0.58″
$6.14 \cdot 10^{-3}$	15°	15°	0	0.02″	14° 59′ 55.47″	4.53″	4.55″
$1.47 \cdot 10^{-2}$	20°	20°	0	0.19″	19° 59′ 42.27″	19.73″	19.92″
$3.00 \cdot 10^{-2}$	25°	25° 0.01″	0.01″	0.95″	24° 58′ 57.14″	1′ 2.86″	1′ 3.82″
$5.36 \cdot 10^{-2}$	30°	30° 0.06″	0.06″	3.71″	29° 57′ 14.84″	2′ 45.16″	2′ 48.93″
$8.93 \cdot 10^{-2}$	35°	35° 0.25″	0.25″	12.13″	34° 53′ 38.41″	6′ 21.59″	6′ 33.98″
$1.41 \cdot 10^{-1}$	40°	40° 0.92″	0.92″	35.16″	39° 46′ 33.56″	13′ 26.44″	14′ 2.51″
$2.15 \cdot 10^{-1}$	45°	45° 2.90″	2.90″	1′ 33.74″	44° 33′ 19.32″	26′ 40.68″	28′ 17.33″

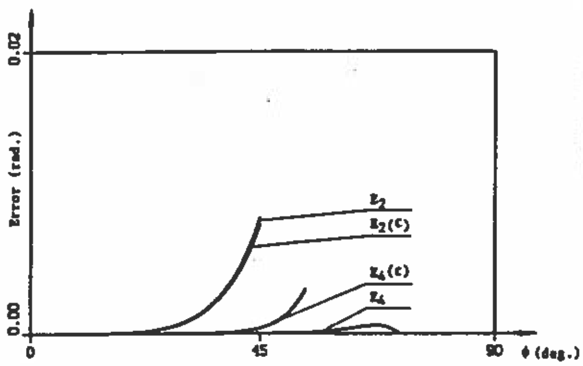

Figure 3 Comparision of the estimated errors and their exact errors of asymptotic solutions $x_2(\varepsilon)$ and $x_4(\varepsilon)$.

$$x(\varepsilon) - \sum_{n=1}^{N} a_n \varepsilon^{(2n-1)/3} \sim a_{N+1}\varepsilon^{(2N+1)/3}, \quad \text{as} \quad \varepsilon \to 0^+, \forall N. \quad (13)$$

Therefore, we can define the asymptotic solution $x_N(\varepsilon)$ and the estimated error $E_N(\varepsilon)$ either by $a_N \varepsilon^{(2N-1)/3}$ (loose), or $a_{N+1}\varepsilon^{(2N+1)/3}$ (precise) as follows.

$$\begin{cases} x_2(\varepsilon) \sim 3^{1/3}\varepsilon^{1/3} - \frac{2}{5}\varepsilon \\ E_2(\varepsilon) \sim \frac{9}{175}3^{2/3}\varepsilon^{5/3} \end{cases} \quad (14a)$$

$$\begin{cases} x_4(\varepsilon) \sim 3^{1/3}\varepsilon^{1/3} - \frac{2}{5}\varepsilon + \frac{9}{175}3^{2/3}\varepsilon^{5/3} - \frac{2}{175}3^{1/3}\varepsilon^{7/3} \\ E_4(\varepsilon) \sim \frac{2}{175}3^{1/3}\varepsilon^{7/3} \end{cases} \quad (15a)$$

or

$$\begin{cases} x_2(\varepsilon) \sim 1.44225\varepsilon^{1/3} - 0.4\varepsilon \\ E_2(\varepsilon) \sim 2.80008\varepsilon^{5/3} \end{cases} \quad (14b)$$

$$\begin{cases} x_4(\varepsilon) \sim 1.44225\varepsilon^{1/3} - 0.4\varepsilon + 0.106976\varepsilon^{5/3} - 0.0164828\varepsilon^{7/3} \\ E_4(\varepsilon) \sim 0.0164828\varepsilon^{7/3} \end{cases} \quad (15b)$$

In Table 1, we compare the asymptotic solutions $x_2(\varepsilon), x_4(\varepsilon)$ and the estimated errors $E_2(\varepsilon), E_4(\varepsilon)$ with their exact values. The exact solutions are obtained by evaluating $\tan x - x = \varepsilon$ directly. From the Table 1, we can see that $E_2(\varepsilon)$ is a good approximation of the exact error; i.e. $a_{N+1}\varepsilon^{(2N+1)/3}$ is a better error expression than $a_N\varepsilon^{(2N-1)/3}$ for $x(\varepsilon) - \sum_{n=1}^{N} a_n\varepsilon^{(2n-1)/3}$; $x_2(\varepsilon)$ is less than the exact x_{exact}, and $x_4(\varepsilon)$ is larger than x_{exact}; i.e.

$$x_2(\varepsilon) < x_{exact} < x_4(\varepsilon).$$

In Table 1, we define

$$E_{exact\ error} = |\text{exact inv}^{-1}(\varepsilon) - \text{asymptotic inv}^{-1}(\varepsilon)|.$$

In Fig.3, both exact errors and estimated errors are shown. We can see that the asymptotic solution $x_2(\varepsilon)$ is a good approximation when $|\varepsilon| < 0.05$, corresponding to $\phi = 30°$, and $x_4(\varepsilon)$ matches the exact solution well. By using expressions for estimated errors $E_2(\varepsilon)$ and $E_4(\varepsilon)$, one can predict the errors a priori, i.e. before calculating the asymptotic solutions.

Note that the errors of asymptotic solutions are less than 1.0% and 0.0018% for $x_2(\varepsilon)$ and $x_4(\varepsilon)$, respectively, when $\phi < \pi/4$, and the maximum error occurs at $\phi = \pi/4$; these percent errors are the maximum error in ϕ expressed as a percentage of the range of ϕ. This accuracy is satisfactory for tooth geometry calculations.

III. Applications

Examples of application of the asymptotic inverse involute function in gearing now follow. The first example is for standard involute gears; the second and third are about gearing system with non-standard operating presure angle and operating distance.

Example 1 A tooth is 1.57 inch thick at the pitch radius of 16 inch and a presure angle of 20°. At what radius does the tooth become pointed ? (From Ref.[5] on p.281)

Let the presure angle and radius be ϕ_x and r_x, respectively, at the point where the tooth becomes pointed. At pitch radius, $t = 1.57$ in, $r = 16$ in, $\phi = 20°$, and according to Ref[5–8]:

$$t_x = 2r_x\left(\frac{t}{2r} + \text{inv}\phi - \text{inv}\phi_x\right) = 0.$$

Then

$$\text{inv}\phi_s = \frac{t}{2r} + \text{inv}\phi = 0.063966883.$$

Hence

$$\phi_s = \text{inv}^{-1}(0.063966883).$$

Note that $\varepsilon = 0.063966883 \ll 1$, using Eq.(14), we get the presure angle

$$\phi_s = (3\varepsilon)^{\frac{1}{3}} - \frac{2}{5}\varepsilon \qquad \text{(radians)}$$
$$= 31.58221025^\circ .$$

The radius is

$$r_s = \frac{r\cos\phi}{\cos\phi_s} = 17.6490971 \text{ in.}$$

with 0.0657% error. The exact values are $\phi_s = 31.64334714^\circ$, and $r_s = 17.66069231$ in, which were obtained by computer using Muller's method to solve the nonlinear equation $\tan\phi_s - \phi_s = 0.063966883$. The values obtained by four term asymptotic series Eq.(15) are $\phi_s = 31.64337431^\circ$, and $r_s = 17.66069747$ in (with $2.92 * 10^{-5}$% error).

Example 2 Two spur gears of 12 and 15 teeth, respectively, are to be cut by a 20° full-depth 6-pitch hob. Determine the center distance at which to generate the gears to avoid undercutting. (From Ref.[6] on p.183)

$$e_1 = \frac{1}{P_d}(k - \frac{N_1}{2}\sin^2\phi) = 0.04968 \text{ in.}$$

$$e_2 = \frac{1}{P_d}(k - \frac{N_2}{2}\sin^2\phi) = 0.02045 \text{ in.}$$

$$\text{inv}\phi' = \text{inv}\phi + \frac{2P_d(e_1 + e_2)\tan\phi}{N_1 + N_2} = 0.02624$$

Using Eq.(14), we get

$$\phi' = \text{inv}^{-1}(0.02624)$$
$$= (3 * 0.02624)^{\frac{1}{3}} - \frac{2}{5} * 0.02624 \qquad \text{(rad.)}$$
$$= 23.95424862^\circ .$$
$$r_1' = \frac{r_1\cos\phi}{\cos\phi'} = \frac{1 * \cos 20^\circ}{\cos\phi'} = 1.028256395 \text{ in.}$$
$$r_2' = \frac{r_2\cos\phi}{\cos\phi'} = \frac{1.25 * \cos 20^\circ}{\cos\phi'} = 1.285320494 \text{ in.}$$
$$C' = r_1' + r_2' = 2.313576889 \text{ in.}$$

with 0.0108634% error in C'. The exact values obtained by using computer are $\phi' = 23.9682549^\circ$, and $C' = 2.313828251$ in. The values obtained by Eq.(15) are $\phi' = 23.9682573^\circ$, and $C' = 2.313828294$ in (with $1.8 * 10^{-6}$% error). The values in Ref.[6] are $\phi' = 23.97^\circ$, and $C' = 2.3144$ in (with 0.02471% error).

Example 3 Two spur gears of 32 and 48 teeth cut by an 8-pitch, 20° pinion cutter mesh together without backlash at the standard center distance of 5 in. To change the speed ratio, it is necessary to replace the 32-tooth pinion with one of 31 teeth. The tooth thickness on the cutting pitch circle of 48-tooth gear and the 5-in. center distance are to remain unchanged. Determine the value of e_1 that will give teeth of the proper thickness to mesh with the 48-tooth gear. The pitch diameter of the pinion cutter D_c is 3.000 in., and the number of teeth in the cutter N_c is 24. (From Ref.[6] on p.196)

$$r_1 = \frac{N_1}{2P_d} = \frac{31}{2 * 8} = 1.9375 \text{ in.}$$
$$r_2 = \frac{N_2}{2P_d} = \frac{48}{2 * 8} = 3.0000 \text{ in.}$$

$$C = \frac{N_1 + N_2}{2P_d} = \frac{31 + 38}{2 * 8} = 4.938 \text{ in.}$$
$$C' = 5.000 \text{ in.}$$
$$\cos\phi' = \frac{C\cos\phi_c}{C'} = \frac{4.938\cos 20^\circ}{5.000}$$
$$\phi' = 21.87^\circ = \phi_r$$

Because $e_2 = 0$, the generating presure angle of the gear 2 is $\phi_{g_2} = 20^\circ$, and we can solve for ϕ_{g_1} as follows:

$$(N_1 + N_c)\text{inv}\phi_{g_1} + (N_2 + N_c)\text{inv}\phi_{g_2} = 2N_c\,\text{inv}\phi_c + (N_1 + N_2)\text{inv}\phi_r .$$

$$(31 + 24)\text{inv}\phi_{g_1} + (48 + 24)\text{inv}\phi_{g_2}$$
$$= 2 * 24\,\text{inv}20^\circ + (31 + 48)\text{inv}21.87^\circ .$$

Therefore,

$$\text{inv}\phi_{g_1} = 0.021773.$$

Using Eq.(14), we get

$$\phi_{g_1} = \text{inv}^{-1}(0.021773)$$
$$= (3 * 0.021773)^{\frac{1}{3}} - \frac{2}{5} * 0.021773 \qquad \text{(rad.)}$$
$$= 22.57568606^\circ .$$
$$e_1 = \frac{(N_1 + N_2)p_b}{2\pi\cos\phi_{g_1}} - C_{std},$$

where C_{std} is the standard center distance between gear 1 and the cutter.

$$p_b = p\cos\phi_c = \frac{\pi}{8}\cos 20^\circ = 0.3690 \text{ in.}$$
$$C_{std} = \frac{N_1 + N_c}{2P_d} = \frac{31 + 24}{2 * 8} = 3.4375 \text{ in.}$$
$$e_1 = \frac{(31 + 24)(0.3690)}{2\pi\cos 22.57568606^\circ} - 3.4375$$
$$= 0.060597545 \text{ in.}$$

with 0.4290168% error. The exact values obtained by using the computer are $\phi_{g_1} = 22.58596534^\circ$, and $e_1 = 0.060858547$ in. The values obtained by Eq.(15) are $\phi_{g_1} = 22.58596675^\circ$, and $e_1 = 0.060858583$ in (with $5.91 * 10^{-5}$% error). The values in Ref.[6] are $\phi_{g_1} = 22.59^\circ$, and $e_1 = 0.06096$ in (with 0.1667% error).

In the above examples, in order to get the values of $\phi = \text{inv}^{-1}(\varepsilon)$, the interpolation of ϕ from an extensive table of $(\phi, \text{inv}\phi)$ is imperative in Ref.[5-8].

Conclusions

Comparisons of the above results with those in Ref.[6] show that the accuracy using the two term asymptotic inverse involute series Eq.(14) is almost the same as the accuracy using interpolation of ϕ from a table. The exact values are almost same as those obtained by the four term asymptotic series Eq.(15) and the latter are much more accurate than the interpolation from a table. Therefore, the use of the asymptotic inverse involute series derived in this paper, instead of an extensive table of $(\phi, \text{inv}\phi)$, are suggested in tooth geometry calculations of the involute gears.

Since the asymptotic series renders the implicit inverse involute function explicit, it could be used to simplify the determination of pinion-cutter offsets required to produce nonstandard spur gears with teeth of equal strength [6,11]. It is expected that this asymptotic series will find its applications not only in the gear tooth geometry calculations, but also in some other fields of engineering where the inverse involute function is involved.

For more information about inverse involute function, economization of inverse involute asymptotic series Eq.(10) by Chebyshev polynomials, and derivation details of inverse involute asymptotic series when $|\varepsilon| \sim 1$ and $|\varepsilon| \gg 1$, one can see Ref.[1].

Acknowledgement

Thanks are due to Prof. K. C. Gupta for his supervising this paper, and to Prof. F. L. Litvin and Prof. F. B. Hanson for their excellent lectures on the theory of gearing and singular perturbations, respectively.

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