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An Historical Note on Finite Rotations

It is shown in this paper that Euler was first to derive the finite rotation formula which is often erroneously attributed to Rodrigues, while Rodrigues was responsible for the derivation of the composition formulae for successive finite rotations and the so-called Euler parameters of finite rotation. Therefore, based upon historical facts, the following nomenclature is suggested: Euler's finite rotation formula, Rodrigues' composition formulae of finite rotations, and Euler-Rodrigues parameters. The text of the paper contains modern symbols and formula forms, while the Appendices contain brief summaries from relevant historical sources with minor alterations in symbols at the most.

I Introduction

To determine the final displacement of a point of a rigid body which undergoes a finite rotation around a fixed axis in space is a classical problem, which has attracted the attention of many researchers. Derivations of the finite rotation formula by using the different tools, such as scalars, quaternions, vectors, matrices, and tensors, etc., have been published. However, there exists some confusion about who was the first contributor of the derivation of the finite rotation formula (Goldstein, 1980, p. 165). In our opinion, the formula should be ascribed to the person who first gave the fundamental formula for finite rotation regardless of the form the formula was written in. Through our historical reference survey, we find that the finite formula is often used without an eponymic designation in textbooks and reference books; but in many cases, the formula is erroneously ascribed to the French mathematician Olinde Rodrigues (1794–1851) such as in Hamel (1949, p. 103), Hiller and Woernle (1984), and Craig (1986, p. 58). Although one can find the derivation of the finite rotation (as well as displacement) formula by using Rodrigues parameters in Rodrigues' paper (Rodrigues, 1840, pp. 403–404), the rotation formula expressed in terms of the direction cosines of the rotation axis and the rotation angle was derived by the Swiss mathematician Leonhard Euler (1707–1783) in Euler (1775b, p. 216) or Euler (Vol. 9, p. 107) and was published 65 years earlier than Rodrigues' paper as is discussed in the following section.

On the other hand, the composition formulae of rotations which are commonly given without an eponymic designation were actually derived by Rodrigues. For proper recognition of contributions, which is also the main purpose of this paper, the composition formulae of rotations should be attributed to Rodrigues.

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II Euler's Finite Rotation Formula and Euler-Rodrigues Parameters

The following is Euler's theorem which is well known: *The general displacement of a rigid body with one point fixed is a rotation about some axis* (Euler, 1775a, p. 202; Euler, Vol. 9, p. 95; Goldstein, 1980, p. 158). The spatial displacement of a rigid body about a fixed point O can always be represented as a spherical rotation of a general point P described by the position vector $\mathbf{r}$ as shown in Fig. 1. According to the Euler's theorem, one can represent the spherical rotation of the vector $\mathbf{r}$ in terms of the rotation parameters: the angle of rotation ϕ and the unit vector $\mathbf{n}$ along the axis of rotation. The following vector representation of the rotation formula is commonly used (Gibbs, 1901, p. 338), (Bisshopp, 1969), (Beatty, 1977), and (Goldstein, 1980, p. 165):

$$\begin{aligned}\mathbf{r}' &= \mathbf{r}\cos\phi + (\mathbf{n}\times\mathbf{r})\sin\phi + \mathbf{n}(\mathbf{n}\cdot\mathbf{r})(1-\cos\phi)\\ &= \mathbf{r} + (\mathbf{n}\times\mathbf{r})\sin\phi + [\mathbf{n}\times(\mathbf{n}\times\mathbf{r})](1-\cos\phi),\end{aligned} \tag{1}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the initial and final positions of the vector, respectively, as is shown in Fig. 1. In order to show the similarity between the formula derived by Euler and other commonly used representations, and to try to explain how Rodrigues has been erroneously credited for the discovery of the formula (1), the following different version of the rotation formula will be discussed.

Since the rotation matrix can be treated as a linear operator, and is intuitively simple, the matrix representation of rigid body rotation is popularly used. If one defines the skew symmetrical matrix $\mathbf{N}$ corresponding to a unit vector $\mathbf{n} = [n_1, n_2, n_3]^T$ as follows

$$\mathbf{N} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}, \tag{2}$$

then the cross product $\mathbf{n}\times\mathbf{r}$ of two vectors $\mathbf{n}$ and $\mathbf{r}$ can be expressed in matrix form as

$$\mathbf{n}\times\mathbf{r}=\mathbf{N}\mathbf{r}. \tag{3}$$

Substituting (3) into (1), we can obtain the following matrix representation of finite rotations

$$\mathbf{r}'=\mathbf{R}\mathbf{r}, \tag{4}$$

$$\mathbf{R}=\mathbf{I}+\mathbf{N}\sin\phi+\mathbf{N}^2(1-\cos\phi), \tag{5}$$

where $\mathbf{I}$ is the unit matrix, and $\mathbf{R}=\mathbf{R}(\phi)$ is the rotation matrix in terms of the direction cosines n_1, n_2, n_3 of the axis of rotation and the rotation angle ϕ.

Without losing generality, one can replace the position vector $\mathbf{r}$ and $\mathbf{r}'$ by the unit vectors $\mathbf{r}_n'$ and $\mathbf{r}_n$, respectively. Then, equation (4) becomes

$$\mathbf{r}_n'=\mathbf{R}\mathbf{r}_n. \tag{6}$$

Defining the direction cosines of unit vectors $\mathbf{n}$, $\mathbf{r}_n$, and r_n', respectively, as follows:

$$\mathbf{n}=\begin{bmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{bmatrix},\ \mathbf{r}_n=\begin{bmatrix}\cos\zeta\\ \cos\eta\\ \cos\theta\end{bmatrix},\ \mathbf{r}_n'=\begin{bmatrix}\cos\zeta'\\ \cos\eta'\\ \cos\theta'\end{bmatrix}, \tag{7}$$

and substituting (5) and (7) into (6) will result in

$$\begin{aligned}
\cos\zeta' &= \cos\zeta(\cos^2\alpha+\sin^2\alpha\cos\phi)\\
&+\cos\eta(\cos\alpha\cos\beta(1-\cos\phi)-\cos\gamma\sin\phi)\\
&+\cos\theta(\cos\alpha\cos\gamma(1-\cos\phi)+\cos\beta\sin\phi)\\
\cos\eta' &= \cos\eta(\cos^2\beta+\sin^2\beta\cos\phi)\\
&+\cos\theta(\cos\beta\cos\gamma(1-\cos\phi)-\cos\alpha\sin\phi)\\
&+\cos\zeta(\cos\alpha\cos\beta(1-\cos\phi)+\cos\gamma\sin\phi)\\
\cos\theta' &= \cos\theta(\cos^2\gamma+\sin^2\gamma\cos\phi)\\
&+\cos\zeta(\cos\alpha\cos\gamma(1-\cos\phi)-\cos\beta\sin\phi)\\
&+\cos\eta(\cos\beta\cos\gamma(1-\cos\phi)+\cos\alpha\sin\phi).
\end{aligned} \tag{8}$$

This form of rotation formulae was actually derived by Euler (1775b, p.216) or Euler (Vol. 9, p. 107) as is shown in Appendix [A.I.]. Therefore, the finite rotation formula (1) or (4) and (5) is nothing but a vector or matrix form of the formulae (8) obtained by Euler more than two hundred years ago. Our conclusion also seems to be supported by Cayley (1846), Benedikt (1944), and Hill (1945).

If one uses Euler angles, ϕ, θ, and ψ, the following rotation matrix can be obtained (Euler, 17??, p. 51; Euler, Vol. 9, p. 424; Goldstein, 1980, p. 147, Eq. (9)).

$$\mathbf{R}=\begin{bmatrix}\cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi-\cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi\\ \cos\psi\sin\phi+\cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi+\cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi\\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta\end{bmatrix}, \tag{9}$$

where the angle ϕ, θ, and ψ are precession, nutation, and spin angles, respectively. It is noticed that the original paper about the derivation details of Euler angles was published in Euler (17??) or Euler (Vol. 9, p. 413–441) after Euler's death (1707–1783) instead of in Euler (1775a), as was erroneously referred to by Whittaker (1937, p. 9). In Euler (1775a), Euler only derived the rotation formula expressed in terms of three independent angles as is shown in Appendix C, but these angles are not what came to be known as Euler angles.

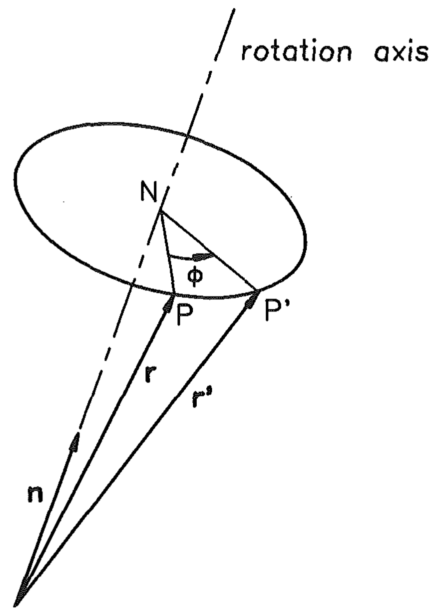

Fig. 1 The vector diagram for Euler's finite rotation formula

Due to the large number of trigonometric functions involved, the Euler angles are difficult to use in numerical computations for the large scale multibody mechanical systems. If one uses Rodrigues parameters (Rodrigues, 1840, p. 400; Bottema and Roth, 1979, p. 148):

$$b_1=n_1\tan\frac{\phi}{2},\ b_2=n_2\tan\frac{\phi}{2},\ b_3=n_3\tan\frac{\phi}{2}, \tag{10}$$

then the expression (1) becomes

$$\mathbf{r}'-\mathbf{r}=\frac{2(\mathbf{b}\times\mathbf{r})+2\mathbf{b}(\mathbf{b}\cdot\mathbf{r})-2(\mathbf{b}\cdot\mathbf{b})\mathbf{r}}{1+\mathbf{b}\cdot\mathbf{b}} \tag{11a}$$

where $\mathbf{b}=[b_1, b_2, b_3]^T$, and the rotation matrix (5) becomes

$$\mathbf{R}=\frac{1}{1+b_1^2+b_2^2+b_3^2}\begin{bmatrix}1+b_1^2-b_2^2-b_3^2 & 2(b_1b_2-b_3) & 2(b_1b_3+b_2)\\ 2(b_1b_2+b_3) & 1-b_1^2+b_2^2-b_3^2 & 2(b_2b_3-b_1)\\ 2(b_1b_3-b_2) & 2(b_2b_3+b_1) & 1-b_1^2-b_2^2+b_3^2\end{bmatrix}. \tag{11b}$$

The corresponding rotational elements derived by Rodrigues are shown in Appendix AII. Rodrigues parameters (10) were used by the Irish mathematician Arthur Cayley (1821–1895) to study the motion of a rigid body in Cayley (1843), (1846), and (1848). One disadvantage of using Rodrigues parameters (10) is that the rotation matrix (11b) is singular when $\phi=\pi$.

For the general computer implementation of multibody dynamic systems, the following four parameters are extensively used:

$$e_0=\cos\frac{\phi}{2},\ e_1=n_1\sin\frac{\phi}{2},\ e_2=n_2\sin\frac{\phi}{2},\ e_3=n_3\sin\frac{\phi}{2}, \tag{12}$$

which satisfy the relation

$$e_0^2+e_1^2+e_2^2+e_3^2=1. \tag{13}$$

These four parameters are customarily called Euler parameters (Whittaker, 1937, p. 8; Bottema and Roth, 1979, p. 150; Goldstein, 1980, p. 153; Kane et al., 1983, p. 12). Presumably, these four parameters were discussed in Euler's paper (1775b, p. 217), as was mentioned by Klein (1884 or 1914, p. 38) and Whittaker (1937, p. 8). The German mathematician Felix Klein's (1849–1925) statements may explain why Euler's name is ascribed to the parameters (12). In Klein (1914, p. 38), Klein claimed that "That it was proper, in the treatment of rotations around a fixed point, to introduce the parameters a, b, c, d of the preceding paragraph (or at least their quotients a/d, b/d, c/d), *Euler* had already found" in Euler (1775b, p. 217). The parameters a, b, c, and d in the above Klein's statements are the parameters (12) e_1, e_2, e_3, and e_0, respectively, and the parameters a/d, b/d, c/d are the so-called Rodrigues' parameters (10). But the fact is that Euler did not use these parameters to describe finite rotations explicitly in Euler (1775b, p. 217). Even half angles of the rotation which are crucial to these parameters can not be found in all of Euler's papers (Nos. 12–16) referred at the end of this paper through our historical reference survey. It was Rodrigues who defined the parameters (12) explicitly and used them to derived the composition formulae in Rodrigues (1840, p. 408) as is shown in Appendix B. Therefore, the parameters (12) should be attributed to Rodrigues instead of Euler. However, to distinguish these from Rodrigues parameters (10), the parameters (12) are called Euler-Rodrigues parameters hereafter, as is also proposed by Altmann (1986, p. 20).

If one uses Euler-Rodrigues parameters (12) and the result derived by Rodrigues (1840, p. 404), and notices that

$$\mathbf{r} = [\mathbf{R}(\phi)]^{-1}\mathbf{r}' = [\mathbf{R}(-\phi)]\mathbf{r}', \quad \mathbf{R}(\phi) = [\mathbf{R}(-\phi)]^{-1}, \quad (14)$$

where $\mathbf{r} = [x, y, z]^T$, and $\mathbf{r}' = [x', y', z']^T = [x+\Delta x, y+\Delta y, z+\Delta z]^T$ as defined in Appendix AII then, the following rotation matrix will be obtained.

$$\mathbf{R} = \begin{bmatrix} 2(e_0^2+e_1^2)-1 & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & 2(e_0^2+e_2^2)-1 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & 2(e_0^2+e_3^2)-1 \end{bmatrix}. \quad (15)$$

The same result can be derived from equation (5) by using Euler-Rodrigues parameters (12) directly.

One of the applications of Euler-Rodrigues parameters is found in the quaternion representation of the finite rotation formula. The quaternion was conceived by the Irish mathematician and astronomer William Rowan Hamilton (1805–1865) when he was on the way to presiding a meeting of the Royal Irish Academy on Monday, October 16, 1843 (Hamilton, 1844; 1853, p. 48; Graves, 1885, Vol. II, p. 434). The number

$$q = a + bi + cj + dk \quad (16)$$

is called a quaternion with the following properties:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \quad (17)$$

If one defines the following unit quaternion via Euler-Rodrigues parameters,

$$q = e_0 + e_1 i + e_2 j + e_3 k, \quad (18)$$

and the vectors $\mathbf{r}$ and $\mathbf{r}'$, which are the initial and final positions of a vector, respectively,

$$\mathbf{r} = 0 + x_1 i + x_2 j + x_3 k, \quad (19)$$

$$\mathbf{r}' = 0 + x_1' i + x_2' j + x_3' k, \quad (20)$$

then the quaternion representation of the rotation formula will result as Cayley (1845), Hamilton (1853, p. 217; 1899, p. 359), Whittaker (1937, p. 9), and Corben and Stehle (1950, p. 171)

$$\mathbf{r}' = q\mathbf{r}q^{-1}, \quad (21a)$$

or

$$\mathbf{r} = q^{-1}\mathbf{r}'q. \quad (21b)$$

According to Cayley's note in his collected mathematical papers (1889, Vol. I, Note 20, p. 586), the quaternion formulae (21) itself was first derived by Hamilton. However, physical meaning of the quaternion form (21) in the context of finite rotation, utilizing Rodrigues parameters (10), was first discovered by Cayley [Cayley (1843)]. When expanded, the quaternion form (21) leads to formulae about which Cayley asserted in Cayley (1845) that "In fact the formulae are precisely those given for such a transformation by M. Olinde Rodrigues" in Rodrigues (1840) shown in Appendix AII.

III Rodrigues Composition Formulae of Finite Rotations

Euler in Euler (1775b) considered a single finite rotation and obtained the formula shown in the preceding section. Euler did not solve the problem of finding the resultant of successive finite rotations, although Euler's theorem affirms its existence. However, Altmann (1986, p. 19) has asserted that in Euler (1775a) "he considered the composition of two successive affine transformations (translation-rotations) and showed that the orientation of the final axes depends on six angular parameters, of which three can be eliminated algebraically, leaving three parameters only, thus determining a rotation. It must be made clear that Euler's approach is algebraic, not geometrical and that it is not constructive. That is, he does not provide closed expressions to determine the angle and axis of the resultant rotation. Euler, however, is most often credited for the solution of the existential, geometric, and constructive problems regarding the composition of two rotations." But the fact is that Euler (1775a) only discussed the representation of a single finite rotation and showed that it could be expressed by nine, and then six dependent angular parameters, and finally by three independent parameters, as is shown in Appendix C. It should be noted that the formulae (8) and (9) of a single finite rotation derived in Euler (1775b), and (17??), respectively, are much nicer in form than that in Euler (1775a).

It was Rodrigues who considered successive finite rotations (Rodrigues, 1840) and found the expressions for determining the orientation of the resultant axis of rotation and the geometrical value of the resultant angle of rotation from the given angles and axis orientations of the two rotations. Let us denote the Rodrigues parameters of the first rotation as

$$\mathbf{W} = \left[n_1 \tan\frac{\phi}{2},\ n_2\frac{\phi}{2},\ n_3 \tan\frac{\phi}{2}\right]^T = \mathbf{n}\tan\frac{\phi}{2}, \quad (22)$$

which is followed by the second rotation

$$\mathbf{W}' = \left[n'_1 \tan\frac{\phi}{2},\ n_2' \tan\frac{\phi'}{2},\ n_3' \tan\frac{\phi'}{2}\right]^T = \mathbf{n}'\tan\frac{\phi'}{2}, \quad (23)$$

with the resultant rotation being

$$\mathbf{W}'' = \left[n_1'' \tan\frac{\phi''}{2}, n_2'' \tan\frac{\phi''}{2}, n_3'' \tan\frac{\phi''}{2}\right]^T = \mathbf{n}''\tan\frac{\phi''}{2}, \quad (24)$$

where $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{n}''$, and ϕ, ϕ', ϕ'', are the unit vectors along the axes and the rotation angles of the first, second, and resultant rotations, respectively. Rodrigues actually obtained the composition formula for two successive rotations (Rodrigues, 1840, p. 408), shown in Appendix B equation (B.1) which can be written in the following vector form (Gibbs, 1901, p. 345; Paul, 1963; Bisshopp, 1969),

$$\mathbf{W}'' = \frac{\mathbf{W}+\mathbf{W}'-\mathbf{W}\times\mathbf{W}'}{1-\mathbf{W}\cdot\mathbf{W}'} = \mathbf{n}''\tan\frac{\phi''}{2}. \tag{25}$$

The formula (25) appears in the later literature, but without an eponymic designation. Besides the formula (25) Rodrigues (1840, p. 408) also derived the composition formula (Appendix B equation (B.2)) which, by using the so-called Euler-Rodrigues parameters (12), becomes as follows:

$$\begin{cases} e_0'' = e_0e_0' - e_1e_1' - e_2e_2' - e_3e_3' \\ e_1'' = e_0e_1' + e_1e_0' - e_2e_3' + e_3e_2' \\ e_2'' = e_0e_2' + e_1e_3' + e_2e_0' - e_3e_1' \\ e_3'' = e_0e_3' - e_1e_2' + e_2e_1' + e_3e_0' \end{cases} \tag{26}$$

The four Euler-Rodrigues parameters (e_0'', e_1'', e_2'', e_3'') are the resultant of two successive rotations (e_0, e_1, e_2, e_3) and (e_0', e_1', e_2', e_3'). Formula (26) can be written in the vector form as

$$\begin{cases} e_0'' = e_0e_0' - \mathbf{e}\cdot\mathbf{e}' \\ \mathbf{e}'' = e_0\mathbf{e}' + e_0'\mathbf{e} - \mathbf{e}\times\mathbf{e}', \end{cases} \tag{27}$$

where vectors, $\mathbf{e}$, $\mathbf{e}'$, and $\mathbf{e}''$ are defined, respectively, as follows:

$$\mathbf{e} = [e_1e_2e_3]^T, \quad \mathbf{e}' = [e_1'e_2'e_3']^T, \quad \mathbf{e}'' = [e_1''e_2''e_3'']^T. \tag{28}$$

If one defines the following quaternions,

$$q = e_0 + e_1i + e_2j + e_3k, \quad q' = e_0' + e_1'i + e_2'j + e_3'k, \quad q'' = e_0'' + e_1''i + e_2''j + e_3''k, \tag{29}$$

then, according to equation (21*a*),

$$\mathbf{r}'' = q'\mathbf{r}'q'^{-1} = q'q\mathbf{r}q^{-1}q'^{-1} = q''\mathbf{r}q''^{-1}. \tag{30}$$

Therefore,

$$q'' = q'q. \tag{31}$$

Formulae (26) and (31) which constitute the theorem for the multiplication of quaternions were discovered by Hamilton according to his letter to John T. Graves dated October 17 of 1843 which was written just one day after the quaternion was conceived (Hamilton, 1844). Perhaps, Felix Klein's statements in Klein (1914, p. 38) are constructive. "It appears, however, that the formulae of composition (i.e., equation (26)) remained still unknown for a long time, till they were discovered by Rodrigues (Rodrigues, 1840). Hamilton then made the same formulae (Hamilton, 1844, p. 489) the foundation of his calculus of quaternions, without at first recognising their significance for the composition of rotations, which was soon brought to light by Cayley (Cayley, 1843, p. 141)." It is noticed that Cayley's paper was published in 1845, instead of 1843, as is referred to by Klein. Since Rodrigues first obtained the formulae (26), along with full physical meaning for combining rotations, we suggest that the formulae (25), (26), and (27) be called the Rodrigues' composition formulae for finite rotations.

IV Conclusions

From the preceding arguments, we conclude that it was Euler, not Rodrigues, who first derived the scalar form of the finite rotation formula (1). Rodrigues, on the other hand, derived the scalar form of the composition formulae (25), (26), and (27) for successive finite rotations, and was also responsible for the parameters (12).

For the terminology, we suggest that equation (1) be called as Euler's finite rotation formula; parameters (12) as Euler-Rodrigues parameters; and equations (25), (26), and (27) as Rodrigues' composition formulae for finite rotations.

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APPENDIX A

Euler's Rotation Formula

A.I. The Rotation Formula Derived By Euler. After defining the three orthogonal coordinates ABC with the origin at a fixed point I as shown in Fig. 2, Euler obtained the following relationship between the direction cosines $\cos\zeta'$, $\cos\eta'$, $\cos\theta'$ of the unit vector Iz (after rotation) and the direction cosines $\cos\zeta$, $\cos\eta$, $\cos\theta$ of the unit vector IZ (before rotation) as follows (Euler 1775b, p. 216; or Vol. 9, p. 107):

$$\begin{aligned}
\cos\zeta' &= \cos\zeta(\cos^2\alpha+\sin^2\alpha\cos\phi)\\
&+ \cos\eta(\cos\alpha\cos\beta(1-\cos\phi)-\cos\gamma\sin\phi)\\
&+ \cos\theta(\cos\alpha\cos\gamma(1-\cos\phi)+\cos\beta\sin\phi)\\
\cos\eta' &= \cos\eta(\cos^2\beta+\sin^2\beta\cos\phi)\\
&+ \cos\theta(\cos\beta\cos\gamma(1-\cos\phi)-\cos\alpha\sin\phi)\\
&+ \cos\zeta(\cos\alpha\cos\beta(1-\cos\phi)+\cos\gamma\sin\phi)\\
\cos\theta' &= \cos\theta(\cos^2\gamma+\sin^2\gamma\cos\phi)\\
&+ \cos\zeta(\cos\alpha\cos\gamma(1-\cos\phi)-\cos\beta\sin\phi)\\
&+ \cos\eta(\cos\beta\cos\gamma(1-\cos\phi)+\cos\alpha\sin\phi),
\end{aligned} \tag{A.1}$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ were the direction cosines of the rotation axis along which the angle ϕ is rotated as is shown in Fig. 2.

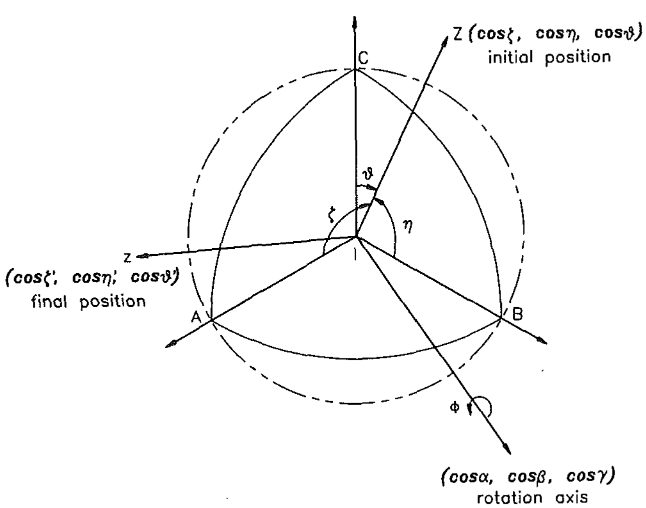

Fig. 2 The schematic diagram for the derivation of the rotation formula by Euler

A.II The Displacement Formulae Derived by Rodrigues[1]. Let the initial and the final coordinates of a point be x, y, z and x', y', z', respectively; the change between the two coordinates is defined by relations

$$x'=x+\Delta x, \quad y'=y+\Delta y, \quad z'=z+\Delta z \tag{A.2}$$

After defining the parameters m, n, and p as follows:

$$m=2\cos\alpha\tan\frac{\phi}{2}, \quad n=2\cos\beta\tan\frac{\phi}{2}, \quad p=2\cos\gamma\tan\frac{\phi}{2}. \tag{A.3}$$

Rodrigues obtained the following formulae (Rodrigues, 1840, p. 404):

$$\begin{aligned}
\Delta x &= u+\frac{py-nz+\frac{m}{2}(mx+ny+pz)-(m^2+n^2+p^2)\frac{x}{2}}{1+\frac{m^2+n^2+p^2}{4}},\\
\Delta y &= v+\frac{mz-px+\frac{n}{2}(mx+ny+pz)-(m^2+n^2+p^2)\frac{y}{2}}{1+\frac{m^2+n^2+p^2}{4}},\\
\Delta z &= w+\frac{nx-my+\frac{p}{2}(mx+ny+pz)-(m^2+n^2+p^2)\frac{z}{2}}{1+\frac{m^2+n^2+p^2}{4}},
\end{aligned} \tag{A.4}$$

where u, v, and w were the three translational terms which were the functions of the rotation angle ϕ and the displacement t along the rotation axis with three direction cosines ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) as is shown in Fig. 3. Equations (A.2) and

[1,2]In the pre-vector era, both left-handed and right-handed coordinate systems were commonly used. Rodrigues and Cayley appear to have utilized the former in their derivations which, for finite rotation formulae, has the effect of reversing the sign of convention for the rotation angle ϕ.

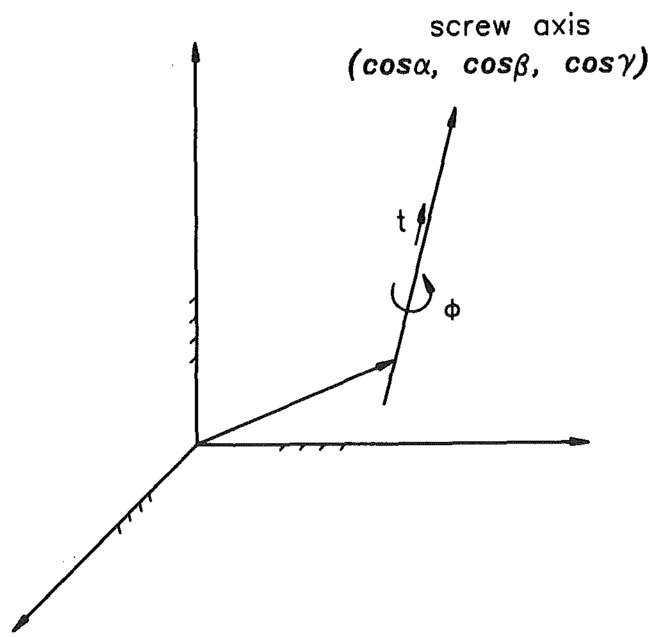

Fig. 3 The schematic diagram for the derivation of the displacement formulae by Rodrigues

(A.4) together formed the displacement formulae from the initial coordinates to the final coordinates.

An alternate form of the displacement formulae derived in Rodrigues (1840, p. 403) was

$$\Delta x = u + \sin\phi(y\cos\gamma - z\cos\beta)$$

$$+ 2\sin^2\frac{1}{2}\phi[\cos\alpha(x\cos\alpha + y\cos\beta + z\cos\gamma) - x],$$

$$\Delta y = v + \sin\phi(z\cos\alpha - x\cos\gamma)$$

$$+ 2\sin^2\frac{1}{2}\phi[\cos\beta(x\cos\alpha + y\cos\beta + z\cos\gamma) - y],$$

$$\Delta z = w + \sin\phi(x\cos\beta - y\cos\alpha)$$

$$+ 2\sin^2\frac{1}{2}\phi[\cos\gamma(x\cos\alpha + y\cos\beta + z\cos\gamma) - z], \qquad (A.5)$$

There are striking similarities among the rotational elements of formula (A.5) and formula (A.1).

APPENDIX B

Rodrigues' Composition Formulae of Finite Rotations[2]

Rodrigues obtained the composition formulae for the rotation from the initial coordinates to the final coordinates for two successive rotations as follows (Rodrigues, 1840, p. 408):

$$\tan\frac{1}{2}\Phi\cos A = \frac{\tan\frac{1}{2}\phi\cos\alpha + \tan\frac{1}{2}\phi'\cos\alpha' + \tan\frac{1}{2}\phi\tan\frac{1}{2}\phi'(\cos\beta\cos\gamma' - \cos\gamma\cos\beta')}{1 - \tan\frac{1}{2}\phi\tan\frac{1}{2}\phi'\cos\nu},$$

$$\tan\frac{1}{2}\Phi\cos B = \frac{\tan\frac{1}{2}\phi\cos\beta + \tan\frac{1}{2}\phi'\cos\beta' + \tan\frac{1}{2}\phi\tan\frac{1}{2}\phi'(\cos\gamma\cos\alpha' - \cos\alpha\cos\gamma')}{1 - \tan\frac{1}{2}\phi\tan\frac{1}{2}\phi'\cos\nu},$$

$$\tan\frac{1}{2}\Phi\cos\Gamma = \frac{\tan\frac{1}{2}\phi\cos\gamma + \tan\frac{1}{2}\phi'\cos\gamma' + \tan\frac{1}{2}\phi\tan\frac{1}{2}\phi'(\cos\alpha\cos\beta' - \cos\beta\cos\alpha')}{1 - \tan\frac{1}{2}\phi\tan\frac{1}{2}\phi'\cos\nu},$$

$$\cos\nu = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma', \qquad (B.1)$$

where $(\cos\alpha, \cos\beta, \cos\gamma)$, $(\cos\alpha', \cos\beta', \cos\gamma')$, and $(\cos A, \cos B, \cos\Gamma)$ were the direction cosines of the first, second, and the resultant rotations with rotation angles ϕ, ϕ', and Φ, respectively.

The composition formulae which satisfy the quaternion multiplication rule was derived as follows (Rodrigues, 1840, p. 408):

$$\cos\frac{1}{2}\Phi = \cos\frac{1}{2}\phi\cos\frac{1}{2}\phi' - \sin\frac{1}{2}\phi'\sin\frac{1}{2}\phi'\cos\nu,$$

$$\sin\frac{1}{2}\Phi\cos A = \sin\frac{1}{2}\phi\cos\frac{1}{2}\phi'\cos\alpha + \sin\frac{1}{2}\phi'\cos\frac{1}{2}\phi\cos\alpha'$$

$$+ \sin\frac{1}{2}\phi\sin\frac{1}{2}\phi'(\cos\beta\cos\gamma' - \cos\gamma\cos\beta'),$$

$$\sin\frac{1}{2}\Phi\cos B = \sin\frac{1}{2}\phi\cos\frac{1}{2}\phi'\cos\beta + \sin\frac{1}{2}\phi'\cos\frac{1}{2}\phi\cos\beta'$$

$$+ \sin\frac{1}{2}\phi\sin\frac{1}{2}\phi'(\cos\gamma\cos\alpha' - \cos\alpha\cos\gamma'),$$

$$\sin\frac{1}{2}\Phi\cos\Gamma = \sin\frac{1}{2}\phi\cos\frac{1}{2}\phi'\cos\gamma + \sin\frac{1}{2}\phi'\cos\frac{1}{2}\phi\cos\gamma'$$

$$+ \sin\frac{1}{2}\phi\sin\frac{1}{2}\phi'(\cos\alpha\cos\beta' - \cos\beta\cos\alpha'), \qquad (B.2)$$

APPENDIX C

Euler's Displacement Transformation Formula

Euler derived the displacement transformation formula in terms of three translation parameters and three independent angular parameters, though not Euler angles, in Euler (1775a; or Vol. 9, p. 84–98). In Fig. 4, a point Z was defined by (p, q, r) with respect to the body coordinates ABC, and the same point z in the fixed coordinates could be expressed in terms of three translational parameters f, g, and h of the origin of the body coordinates ABC and the body coordinates p, q, and r via the rotational parameters F, F', F''; G, G', G''; and H, H', H''.

$$x = f + Fp + F'q + F''r,$$

$$y = g + Gp + G'q + G''r, \qquad (C.1)$$

$$z = h + Hp + H''q + H''r$$

Substituting the point $Z = (p, 0, 0)$ in the body coordinates ABC into equation (C.1) and considering the condition for the rigid body

$$|IZ| = |iz|, \qquad (C.2)$$

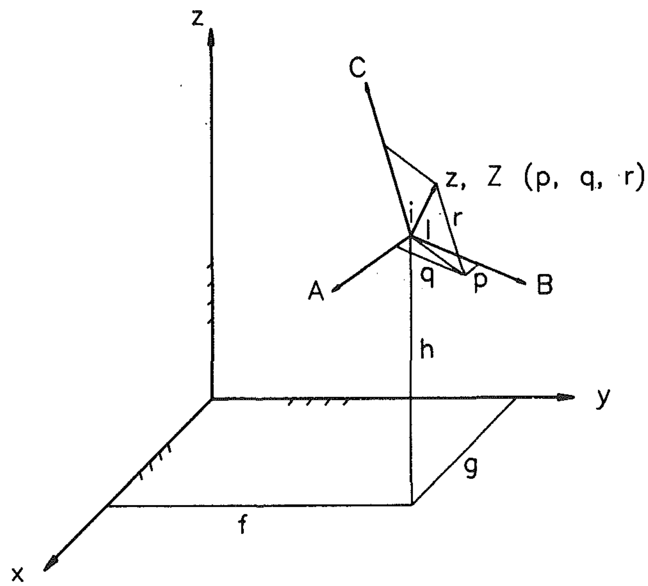

Fig. 4 The schematic diagram for the derivation of the displacement transformation formula by Euler

$$p^2=(x-f)^2+(y-g)^2+(z-h)^2=p^2(F^2+G^2+H^2), \quad (C.3)$$

one would obtain

$$F^2+G^2+H^2=1. \quad (C.4a)$$

Similarly, one could derive

$$F''^2+G'^2+H'^2=1, \quad (C.4b)$$

$$F''^2+G''^2+H''^2=1. \quad (C.4c)$$

Equation (C.4a) would be satisfied after defining

$$F=\sin\zeta, \quad G=\cos\zeta\sin\eta, \quad H=\cos\zeta\cos\eta. \quad (C.5a)$$

Similarly, F', G', H'; and F'', G'', H'' were defined as

$$F'=\sin\zeta', \; G''=\cos\zeta'\sin\eta', \; H'=\cos\zeta'\cos\eta', \quad (C.5b)$$

$$F''=\sin\zeta'', \; G''=\cos\zeta''\sin\eta'', \; H''=\cos\zeta''\cos\eta'', \quad (C.5c)$$

which satisfy equations ($C.4b$), ($C.4c$). In order to reduce the six dependent parameters, ζ, η, ζ', η', ζ'', and η'' to three independent parameters, substituting the point $Z=(p, q, 0)$ into equations ($C.1$) and using the conditions ($C.2$) and ($C.4$) would result

$$p^2+q^2=p^2+q^2+2pq(FF'+GG'+HH'). \quad (C.6)$$

Therefore,

$$FF'+GG'+HH'=0. \quad (C.7)$$

Similarly,

$$FF''+GG''+HH''=0, \quad (C.7b)$$

$$F'F''+G'G''+H'H''=0. \quad (C.7c)$$

After substituting F's, G's, and H's in equation ($C.5$) into ($C.7$), one could get

$$\tan\zeta\tan\zeta'=-\cos(\eta-\eta'),$$
$$\tan\zeta'\tan\zeta''=-\cos(\eta'-\eta''), \quad (C.8)$$
$$\tan\zeta''\tan\zeta=-\cos(\eta''-\eta).$$

or,

$$\tan\zeta=-\frac{\Delta}{\cos(\eta'-\eta'')}, \quad \tan\zeta'=-\frac{\Delta}{\cos(\eta''-\eta)},$$
$$\tan\zeta''=-\frac{\Delta}{\cos(\eta-\eta')}, \quad (C.9)$$

where Δ was defined as

$$\Delta=\tan\zeta\tan\zeta'\tan\zeta''=\sqrt{-\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta)}. \quad (C.10)$$

Through equation ($C.9$), one could obtain

$$\sin\zeta=-\sqrt{\cot\theta'\cot\theta''}, \quad \cos\zeta=\sqrt{1-\cot\theta'\cot\theta''}$$
$$\sin\zeta'=-\sqrt{\cot\theta''\cot\theta}, \quad \cos\zeta'=\sqrt{1-\cot\theta''\cot\theta} \quad (C.11)$$
$$\sin\zeta''=-\sqrt{\cot\theta\cot\theta'}, \quad \cos\zeta''=\sqrt{1-\cot\theta\cot\theta'}$$

where θ, θ', and θ'' were defined as

$$\theta=\eta'-\eta'', \quad \theta'=\eta''-\eta, \quad \theta''=\eta-\eta', \quad (C.12)$$

which satisfy the relation

$$\theta+\theta'+\theta''=0. \quad (C.13)$$

Defining

$$t=\cos\theta, \quad t'=\cot\theta', \quad t''=\cot\theta'' \quad (C.14)$$

would finally bring equation ($C.1$) into

$$x=f-\sqrt{t't''}p+\sin\eta\sqrt{1-t't''}q+\cos\eta\sqrt{1-t't''}r,$$

$$y=g-\sqrt{tt''}p+\sin\eta'\sqrt{1-t''}tq+\cos\eta\sqrt{1-t''}tr, \quad (C.15)$$

$$z=h-\sqrt{tt'}p+\sin\eta''\sqrt{1-tt'}q+\cos\eta''\sqrt{1-tt''}r,$$

where, t, t', and t'' were the functions of three independent rotational parameters η, η', and η'' expressed by equation ($C.12$) and ($C.14$).

The trigonometrical identity

$$\tan\theta''=-\tan(\theta+\theta')=-\frac{\tan\theta+\tan\theta'}{1-\tan\theta\tan\theta'}, \quad (C.16)$$

gives

$$tt''+t't''+tt'=1, \quad (C.17)$$

which would result in the orthogonality relations ($C.4$), ($C.7$) among the coefficients of p, q, and r in equation ($C.15$).